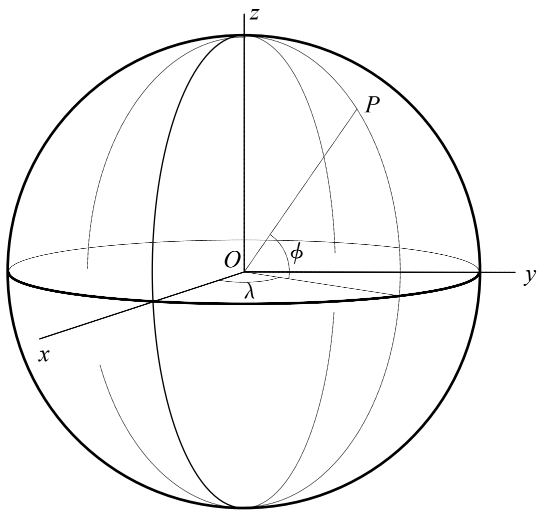
Define a 3-dimensional rectangular cartesian coordinate system with its origin O at the centre of the Earth, and the equator lying in the xy-plane, and let the x-axis pass through the point where the Greenwhich meridian intersects the equator. The angles ϕ, λ in the diagram are the latitude and longitude respectively of the point P and the cartesian coordinates of P will be



$$\left.\begin{aligned} x &= R\cos\phi\cos\lambda \\ y &= R\cos\phi\sin\lambda \\ z &= R\sin\phi \end{aligned}\right\}$$

where R is the Earth's radius, so that the position-vector of P is

$$\mathbf{P} = \overrightarrow{OP} = R\cos\phi\cos\lambda\,\mathbf{i} + R\cos\phi\sin\lambda\,\mathbf{j} + R\sin\lambda\,\mathbf{k}.$$

If N' and S' are the north and south magnetic poles, then by simple vector algebra

$$\overrightarrow{OS'} + \overrightarrow{S'N'} = \overrightarrow{ON'}$$

i.e.

$$\overrightarrow{S'N'} = \overrightarrow{ON'} - \overrightarrow{OS'}$$

$$= R(\cos\phi_1\cos\lambda_1 - \cos\phi_2\cos\lambda_2)\mathbf{i} + R(\cos\phi_1\sin\lambda_1 - \cos\phi_2\sin\lambda_2)\mathbf{j} + R(\sin\phi_1 - \sin\phi_2)\mathbf{k},$$

where (ϕ_1,λ_1) and (ϕ_2,λ_2) are the geodetic positions (i.e. the latitudes and longitudes) of the north and south magnetic poles, respectively. The magnitude of this vector is

$$|\overrightarrow{S'N'}| = R\sqrt{(\cos\phi_1\cos\lambda_1 - \cos\phi_2\cos\lambda_2)^2 + (\cos\phi_1\sin\lambda_1 - \cos\phi_2\sin\lambda_2)^2 + (\sin\phi_1 - \sin\phi_2)^2}$$

$$= R\sqrt{2\{1 - \sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2\cos(\lambda_1 - \lambda_2)\}}$$

and, to find the angle θ between the line joining the magnetic poles and the axis of rotation, we must form the scalar product of this vector with the unit vector **k** (which coincides with the axis of rotation). We obtain

$$R(\sin\phi_1 - \sin\phi_2) = |\overrightarrow{S'N'}| \times 1 \times \cos\theta$$

i.e.

$$\cos\theta = \frac{\sin\phi_1 - \sin\phi_2}{\sqrt{2\{1 - \sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2\cos(\lambda_1 - \lambda_2)\}}}$$

Using the positions established by the 2001 surveys, i.e.

$$\phi_1 = +81^\circ{\cdot}3,\ \lambda_1 = -110^\circ{\cdot}8$$

and

$$\phi_2 = -64^\circ{\cdot}7,\ \lambda_2 = +138^\circ{\cdot}0.$$

we find that

$$\cos\theta = \frac{1{\cdot}89258}{\sqrt{3{\cdot}83411}} = \frac{1{\cdot}89258}{1{\cdot}95809} = 0{\cdot}96654$$

and so $\theta = 14^\circ{\cdot}86286 \approx 14^\circ 52'$

Any point P lying on the line joining two other points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) will have coordinates

$$\left.\begin{aligned} x &= x_1 + (x_2 - x_1)u \\ y &= y_1 + (y_2 - y_1)u \\ z &= z_1 + (z_2 - z_1)u \end{aligned}\right\}$$

for some u, and the square of that point's distance from another point P_0 with coordinates (x_0,y_0,z_0) will be

$$s^2 = \{(x_1 - x_0) + (x_2 - x_1)u\}^2 + \{(y_1 - y_0) + (y_2 - y_1)u\}^2 + \{(z_1 - z_0) + (z_2 - z_1)u\}^2.$$

The minimum value of s^2 occurs at the point on P_1P_2 closest to P_0 and so, to find that point, we must solve

$$\frac{\partial}{\partial u}(s^2) = 0$$

for u. Now,

$$\frac{\partial}{\partial u}(s^2) = 2(x_2 - x_1)\{(x_1 - x_0) + (x_2 - x_1)u\} + 2(y_2 - y_1)\{(y_1 - y_0) + (y_2 - y_1)u\} + 2(z_2 - z_1)\{(z_1 - z_0) + (z_2 - z_1)u\}$$

$$= 2\{(x_2 - x_1)(x_1 - x_0) + (y_2 - y_1)(y_1 - y_0) + (z_2 - z_1)(z_1 - z_0)\} + 2\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}u$$

and the point on P_1P_2 closest to P_0 is therefore given by

$$u = -\frac{(x_2 - x_1)(x_1 - x_0) + (y_2 - y_1)(y_1 - y_0) + (z_2 - z_1)(z_1 - z_0)}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and its coordinates will be

$$\left.\begin{aligned} x' &= x_1 - (x_2 - x_1)\frac{(x_2 - x_1)(x_1 - x_0) + (y_2 - y_1)(y_1 - y_0) + (z_2 - z_1)(z_1 - z_0)}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ y' &= y_1 - (y_2 - y_1)\frac{(x_2 - x_1)(x_1 - x_0) + (y_2 - y_1)(y_1 - y_0) + (z_2 - z_1)(z_1 - z_0)}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ z' &= z_1 - (z_2 - z_1)\frac{(x_2 - x_1)(x_1 - x_0) + (y_2 - y_1)(y_1 - y_0) + (z_2 - z_1)(z_1 - z_0)}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \end{aligned}\right\}$$

If we take P_1 and P_2 as the positions of the Earth's magnetic poles in 2001 (i.e. 81°·3N, 110°·8W and 64°·7S, 138°·0E), we find that their cartesian coordinates (in units of the Earth's radius R) were

$x_1 = -0{\cdot}05371$, $y_1 = -0{\cdot}14140$, $z_1 = +0{\cdot}98849$ for the north magnetic pole

and

$x_2 = -0{\cdot}31759$, $y_2 = +0{\cdot}28596$, $z_2 = -0{\cdot}90408$ for the south magnetic pole,

and the point P' on the magnetic axis closest to the centre of the Earth ($x = 0$, $y = 0$, $z = 0$) was

$x' = -0{\cdot}18565$, $y' = +0{\cdot}07228$, $z' = +0{\cdot}04221$ (again in units of the Earth's radius R).

The distance of this point from the centre of the Earth is

$$R\sqrt{(x')^2 + (y')^2 + (z')^2} = 0{\cdot}20365R \approx 1{,}300 \text{ km}.$$